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LETTER TO THE EDITOR

The partition function and level density for Yang–Mills–Higgs quantum mechanics

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Abstract
We calculate the partition function $Z(t)$ and the asymptotic integrated level density $N(E)$ for Yang–Mills–Higgs quantum mechanics for two and three dimensions ($n = 2, 3$). Due to the infinite volume of the phase space Γ on energy shell for $n = 2$, it is not possible to completely disentangle the coupled oscillators (x^2y^2 model) from the Higgs sector. The situation is different for $n = 3$ for which Γ is finite. The transition from order to chaos in these systems is expressed by the corresponding transitions in $Z(t)$ and $N(E)$, analogous to the transitions in adjacent level spacing distribution from Poisson distribution to Wigner–Dyson distribution. We also discuss a related system with quartic coupled oscillators and two dimensional quartic free oscillators for which, contrary to YMHQM, both coupling constants are dimensionless.

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1. Introduction

The discovery of chaoticity in the classical Yang–Mills (YM) equations [1] has attracted attention to the system of coupled quartic oscillators with the potential x^2y^2, where x and y are functions of time only. This system is the simplest limiting case for the homogeneous YM equations (the so-called YM classical mechanics) with $n = 2$ degrees of freedom. Despite its simplicity, the x^2y^2 model exhibits a rich versatile chaotic behaviour and belongs to the most chaotic potential systems known. Not surprisingly, this potential has been used in many fields, including chemistry, astronomy, astrophysics and cosmology (chaotic inflation).

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From the quantum mechanical point of view, this system (YM quantum mechanics) has a discrete spectrum [2, 3] in spite of having an infinite volume of energetically accessible phase space Γ [4, 5]

$$\Gamma = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\delta\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{g^2}{2}x^2y^2 - E\right),$$

where the dot stands for $\mathrm{d}/\mathrm{d}t$. This potential, therefore, violates the Weil's law [6], a semiclassical relation, which states that the average number $N(E)$ of quantum energy levels with energy less than E is asymptotically proportional to Γ.

Some time ago, there appeared two important papers [7, 8] devoted to the calculation of the partition function $Z(t)$ and the asymptotic integrated level density $N(E)$ for the x^2y^2 model. Their method was based on an adiabatic separation in the partition function's dependence on x and y out in the narrow channels of the equipotential surface $|xy| = \text{constant}$. Remarkably the dependence on the boundary dividing the two regions (central ($|x| \leqslant Q$, $|y| \leqslant Q$) and channel ($Q \leqslant |x|, |y| < \infty$)) with quite different physics (essentially classical for the first region and intrinsically quantum for the second region) disappears in the final result for the partition function. This insensitivity to the dependence on the boundary (the value of Q) may seem to bode well for this method of calculation.

However, *a priori* we do not expect that the partition function of a non-integrable system with infinite phase space volume is calculable without any approximation. Therefore, we seek an alternative approach which brings to the fore the fact that $\Gamma = \infty$ for the $n = 2$ case. With this in mind, in this paper we calculate the partition function and the asymptotic integrated level density for the so-called Yang–Mills–Higgs quantum mechanics (YMHQM) [9, 4]. It is interesting to examine how $Z(t)$ and $N(E)$ behave in and limit the Higgs coupling to the x- and y-amplitudes one vanishes and one recovers the pure x^2y^2 system. We also consider the YMHQM for the case of $n = 3$ for which Γ is finite [4, 5] even for the pure YM system.

2. Yang–Mills–Higgs mechanics

There are several mechanisms that can suppress the classical chaoticity of chaotic systems with dimension $n = 2$ and, in particular, of the YM system (see [5]). One of them is the well-known Higgs mechanism. For spatially homogeneous fields, if only the interaction of the YM fields with the Higgs vacuum is considered, the classical Hamiltonian density for $n = 2$ is given by

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{g^2}{2}x^2y^2 + \frac{v^2}{2}(x^2 + y^2). \tag{1}$$

It is known [9] that there is a classical 'phase transition' from chaos to regular motions as the vacuum expectation value of the Higgs field $\langle\phi\rangle = v$ gets large enough. Here there is one control parameter $\kappa = \frac{g^2v^4}{H}$, where H is the conserved energy density. At large κ the motion is regular, whereas at $\kappa = 0$ one deals with the developed chaos of YM classical mechanics. In fact, chaos appears already at $\kappa \approx 0.60$ [9]. Therefore, in the study of the quantum counterpart of equation (1), we expect that there is a transition from one type of $N(E)$ to another, depending on the parameter v. The analogous transition in the adjacent energy level spacing distribution, as a function of v, was predicted [10] and established in several papers [11].

In the next section we calculate the partition function for the quantum-mechanical counterpart of the classical Hamiltonian density equation (1). In section 4, we calculate the integrated level density. Section 5 is devoted to the $n = 3$ case. We conclude in the last section with some discussions involving a related system with potential of the form $\frac{g^2}{2}x^2y^2 + \frac{b^2}{4}(x^4 + y^4)$.

3. Partition function for YMH quantum mechanics

The quantum Hamiltonian corresponding to equation (1) is given by (mass $m = 1$ in the following)

$$\mathcal{H} = \frac{\vec{\mathbf{p}}^2}{2} + \frac{g^2}{2}x^2y^2 + \frac{v^2}{2}(x^2 + y^2) \tag{2}$$

where $\vec{\mathbf{p}} = -\mathrm{i}\hbar\vec{\nabla}$ is the momentum operator. First a word on units. All quantities below are given in units of energy $E : [\mathcal{H}] = E, [t] = E^{-1}, [x] = [y] = E^{1/4}, [g] = E^0, [v] = E^{1/4}, [\hbar] = E^{3/4}$. It is obvious that the operator equation (2) has a discrete spectrum.

Here we calculate the trace of the heat kernel $\exp(-t\mathcal{H})$, the partition function for the Hamiltonian operator. The trace is defined for any quantum operator $\mathcal{A}_W$ in the Wigner representation (i.e. in the classical $2n$-dimensional phase space)

$$\mathrm{Tr}(\mathcal{A}_W) = \frac{1}{(2\pi\hbar)^n}\int \mathrm{d}\vec{\mathbf{p}}\,\mathrm{d}\vec{\mathbf{q}}\,A_W(\vec{\mathbf{p}}, \vec{\mathbf{q}}). \tag{3}$$

Thus, we have

$$Z(t) = \mathrm{Tr}(\mathrm{e}^{-t\mathcal{H}}) = \int_0^\infty \mathrm{d}E\,\mathrm{e}^{-tE}\rho(E). \tag{4}$$

The second equation expresses the fact that the partition function $Z(t)$ and the density of eigenstates $\rho(E)$ form a Laplace transform pair. The asymptotic integrated density of states $N(E)$ is given by the inverse Laplace transform of $Z(t)/t$

$$N(E) = \int_0^E \mathrm{d}E'\,\rho(E') = L^{-1}\left(\frac{Z(t)}{t}\right). \tag{5}$$

Integrating equation (4) over p_x and p_y for the Hamiltonian given by equation (2), we get

$$Z(t) = \frac{1}{(2\pi\hbar)^2}\frac{2\pi}{t}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-t[v^2(x^2+y^2)+g^2x^2y^2]/2}. \tag{6}$$

After the integration over x, we can make a change of variable from y to $s = \frac{1}{2}tv^2y^2$ to obtain

$$Z(t) = \frac{1}{(2\pi\hbar)^2}\left(\frac{2\pi}{t}\right)^{3/2}\sqrt{\frac{2}{tv^4}}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s}s^{-1/2}\left(1 + \frac{s}{2z}\right)^{-1/2} \tag{7}$$

with $z \equiv \frac{tv^4}{4g^2}$. But aside from a factor of $\sqrt{2z}\,\mathrm{e}^z$, the integral is just a representation of $K_0(z)$, the modified Bessel function of the third kind or the MacDonald function of order 0 (see, e.g., p 140 of [12]). Thus, we arrive at the precise expression for the partition function corresponding to the Hamiltonian equation (2)

$$Z(t) = \frac{1}{\sqrt{2\pi}}\frac{1}{g\hbar^2t^{3/2}}\exp\left(\frac{tv^4}{4g^2}\right)K_0\left(\frac{tv^4}{4g^2}\right). \tag{8}$$

So far we have not made any calculational approximations. For two uncoupled oscillators (corresponding to $g = 0$) equation (8) gives (with the aid of $K_0(z) \simeq \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}$ as $z \to \infty$)

$$Z(t) = \frac{1}{(\hbar vt)^2} \tag{9}$$

as it should. For the sake of completeness, let us also give the partition function for one oscillator ($n = 1$)

$$Z(t) = \frac{1}{\hbar v t}. \tag{9'}$$

From equation (8), it is obvious that $Z(t)$ diverges logarithmically in the limit $v \to 0$ (the x^2y^2 model). Indeed, using $K_0(z) \simeq -\log(z/2) - C$ (with C being the Euler constant) as $z \to 0$, we get, for $v \simeq 0$,

$$Z(t) \simeq \frac{1}{\sqrt{2\pi}} \frac{1}{g\hbar^2 t^{3/2}} \left(\log \frac{8g^2}{tv^4} - C \right). \tag{10}$$

The impossibility to disentangle the coupled oscillators from the uncoupled ones is a reflection of the logarithmic divergence of the phase space volume Γ on the E-shell at $v = 0$ for $n = 2$, as we have mentioned above. For $n = 3$ (see section 5), Γ is finite and the precise $v = 0$ limit exists.

4. Integrated level density $N(E)$ ($n = 2$)

It is well known that the asymptotic integrated density of states $N(E)$ is related to the small t divergence of the partition function $Z(t)$, according to the Karamata–Tauberian theorem (see, e.g., [3]). Using equation (8) and equation (5), we obtain the precise expression for $N(E)$ (with the aid of formula 3.16.3.3 of [13])

$$\begin{aligned} N(E) &= \frac{1}{\sqrt{2\pi}} \frac{1}{g\hbar^2} L^{-1} \left(\frac{1}{t^{5/2}} \exp\left(\frac{tv^4}{4g^2} \right) K_0 \left(\frac{tv^4}{4g^2} \right) \right) \\ &= \frac{E^2}{2\hbar^2 v^2} F\left(\frac{1}{2}, \frac{1}{2}; 3; -\frac{2g^2 E}{v^4} \right) \end{aligned} \tag{11}$$

where F is the Gauss hypergeometric function. For $g = 0$ (the case of two uncoupled oscillators) we have, from equation (11),

$$N(E) = \frac{1}{2\hbar^2 v^2} E^2 \tag{12}$$

in agreement with equation (9). For completeness, for one oscillator, we have, from equation (9′),

$$N(E) = \frac{1}{\hbar v} E. \tag{12'}$$

For the $v \to 0$ limit, one needs the asymptotic expression of $F(a, b; c; z)$. Using formula 9.7.7 of [12], we get

$$N(E) \simeq \frac{2\sqrt{2}}{3\pi} \frac{E^{3/2}}{g\hbar^2} \left(\log \frac{g^2 E}{v^4} + 5 \log 2 - \frac{8}{3} \right). \tag{13}$$

One can verify that the $Z(t)$ and $N(E)$ for equation (9), equation (9′), and the corresponding equation (12), equation (12′), and the *ratio* of the logarithmic parts of equations (10) and (13), are in accordance with theorems 1.1 and 1.4 of [3]. Also from equations (12) and (13), we see that the transition from order ($g = 0$) to chaos ($g \neq 0$ and large, v small, $\kappa \neq 0$)[4] corresponds to the change in the E-dependence of $N(E)$ from $N(E) \sim E^2$ to $N(E) \sim E^{3/2} \log E$, analogous to the change in the neighbour level spacing distribution from the Poisson distribution to the

[4] We need to emphasize that, from the results of [9], the chaotic regime for $N(E)$ actually begins at finite $v = (\kappa E/g^2)^{1/4}$.

Wigner–Dyson distribution. It can be seen that as the power α of the homogeneous potential $|xy|^\alpha$ increases from 1 to ∞, there is [3] a systematic decrease in the power of E in $N(E)$ from 2 to 1, the latter corresponding to the case of the hyperbola billiard with $N(E) \sim E \log E$ (see [14]). This observation may be relevant to the Hilbert–Polya–Berry program in identifying a quantum (chaotic) Hamiltonian whose eigenvalues reproduce the Riemann zeta-function zeros.

5. Some remarks on the three-dimensional YM and YMH quantum mechanics

Let us now consider the YMH quantum mechanics for $n = 3$ with the Hamiltonian (with mass $m = 1$ again)

$$\mathcal{H} = \frac{1}{2}\vec{\mathbf{p}}^2 + \frac{g^2}{2}(x^2y^2 + y^2z^2 + z^2x^2) + \frac{v^2}{2}(x^2 + y^2 + z^2). \tag{14}$$

For the partition function, the integrations over $\vec{\mathbf{p}}$ can be easily carried out. Introducing the cylindrical coordinates ($x = r\cos\phi$, $y = r\sin\phi$, $z = z$) and performing the integrations over z and ϕ, we obtain

$$Z(t) = \frac{1}{(2\pi\hbar)^3}\left(\frac{2\pi}{t}\right)^2 2\pi \int_0^\infty \frac{r\,\mathrm{d}r}{\sqrt{v^2 + g^2r^2}} \exp\left(-\frac{tv^2r^2}{2} - \frac{tg^2r^4}{16}\right) I_0\left(\frac{tg^2r^4}{16}\right) \tag{15}$$

where $I_0(z)$ is the modified Bessel function of the first kind of order 0.

We see that the integral in equation (15) is well behaved at $v = 0$; it is equal to

$$\frac{1}{4g}\int_0^\infty \mathrm{d}u \frac{1}{u^{3/4}}\,\mathrm{e}^{-g^2tu/16} I_0\left(\frac{g^2tu}{16}\right). \tag{15$'$}$$

The integration over u can be done by using formula 2.15.3.3 of [15] to yield

$$Z(t) = \frac{\Gamma^3\left(\frac{1}{4}\right)}{2^{7/4}\pi^{3/2}} \frac{1}{g^{3/2}\hbar^3 t^{9/4}} \tag{16}$$

which agrees with equation (4.5) of [8] (after taking into account the extra factor of $\frac{1}{2}$ in the definition of the potential in equation (14)). The difference between the $n = 2$ and $n = 3$ cases is due to the fact that for $n = 3$ the energetically accessible region pinches as $1/x^2$ at large x, in contrast to the $n = 2$ case for which it pinches as $1/x$, leading to the logarithmic divergent result (equation (10)) at $v = 0$. Note that the contribution from the channels ($t^{1/4}x \gg 1$) for $n = 3$ is negligible in the method of separating the central part from the channels used in [7, 8]. Thus, for $n = 3$ the coupled oscillators are disentangled from the free oscillators. From equation (16), it follows that

$$N(E) = \frac{16}{45}\frac{2^{1/4}\Gamma^2\left(\frac{1}{4}\right)}{\pi^{3/2}}\left(\frac{E^{3/4}}{\sqrt{g\hbar}}\right)^3 \tag{17}$$

which agrees with the result given in [8] (the first of equation (5.19)).

Finally, for the sake of completeness, here are the expressions for $Z(t)$ and $N(E)$ at $g = 0$ for the case of $n = 3$. Equation (15) yields

$$Z(t) = \frac{1}{(\hbar vt)^3}. \tag{18}$$

Of course, equations (9), (9$'$) and (18) are in accordance with the well-known relation of the trace of the n-harmonic-oscillator heat kernel, namely, $Z(t) = \left(2\sinh\frac{\hbar vt}{2}\right)^{-n}$ in the quasiclassical approximation ($t \to 0$) [7, 8]. From equation (18) we have, for $N(E)$

$$N(E) = \frac{1}{6\hbar^3v^3}E^3. \tag{19}$$

Equations (16)–(19) show that the $Z(t)$ and $N(E)$ for the $n = 3$ case are in full agreement with theorem 1.1 of [3].

6. Discussions

From the previous two sections, we see that the $\log E$ factor in $N(E)$ is specific to the x^2y^2 model (with or without the Higgs term), and is not generic to chaotic systems with potentials $|xy|^\alpha$ $(\alpha = 2, 3, \ldots)$ for $n > 2$. But one may wonder whether it is generic to other homogeneous potentials for $n = 2$. The answer depends on the type of homogeneous potentials added to the x^2y^2 term. Consider the following example of $n = 2$ system (with finite phase space), described by the following quantum Hamiltonian:

$$\mathcal{H} = \frac{\vec{\mathbf{p}}^2}{2} + \frac{g^2}{2}x^2y^2 + \frac{b^2}{4}(x^4 + y^4). \tag{20}$$

Like g, b is dimensionless. It is known that for $b^2 = g^2/3$ the system equation (20) is integrable [16]. In the calculation of $Z(t)$, the integrations over x and y can best be done by using the polar coordinates. After the Gaussian integration over the radial coordinate, the integration over the angle yields the complete elliptic integral of the first kind [12] which is related to the Gauss hypergeometric function. The result is given by

$$Z(t) = \frac{\sqrt{\pi}}{2}\frac{1}{b\hbar^2 t^{3/2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{b^2 - g^2}{2b^2}\right). \tag{21}$$

The corresponding asymptotic integrated density of states $N(E)$ is given by

$$N(E) = \frac{2E^{3/2}}{3b\hbar^2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{b^2 - g^2}{2b^2}\right). \tag{22}$$

For the $b \to 0$ limit, we use formula 9.7.7 of [12] again to get

$$Z(t) \simeq \frac{1}{\sqrt{2\pi}} \frac{1}{g\hbar^2 t^{3/2}} \log \frac{8g^2}{b^2}. \tag{23}$$

We note the logarithmic divergence as $b \to 0$, i.e. when one tries to disentangle the coupled quartic oscillators from the free quartic oscillators. A simple inverse Laplace transform yields the level density

$$N(E) \simeq \frac{2\sqrt{2}}{3\pi} \frac{E^{3/2}}{g\hbar^2} \log \frac{8g^2}{b^2}. \tag{24}$$

Just as there is no $\log t$ dependence in $Z(t)$, we find no $\log E$ dependence in $N(E)$ for this system. This result is not surprising from the viewpoint of dimensional considerations since both couplings in this example are dimensionless. In the case of YMHQM, the coupling v is dimensionful, and due to this, $\log t$ and $\log E$ appear in equations (10) and (13), respectively, in the $v \to 0$ limit.

For completeness we also give the partition function for the $g = 0$ case, which corresponds to an integrable system. Using $F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}\right) = \Gamma^2\left(\frac{1}{4}\right)\big/(2\pi^{3/2})$, we have

$$Z(t) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\pi} \frac{1}{b\hbar^2 t^{3/2}}. \tag{25}$$

The corresponding asymptotic integrated density of states is given by

$$N(E) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{3\pi^{3/2}} \frac{E^{3/2}}{b\hbar^2}. \tag{26}$$

For another integrable case with $b^2 = g^2/3$ in this example [16], using $F\left(\frac{1}{2}, \frac{1}{2}; 1; -1\right) = \Gamma^2\left(\frac{1}{4}\right)\big/(2\pi)^{3/2}$, we get

$$Z(t) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2^{5/2}\pi}\frac{1}{b\hbar^2 t^{3/2}} \tag{27}$$

and

$$N(E) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{3(2\pi^3)^{1/2}}\frac{E^{3/2}}{b\hbar^2}. \tag{28}$$

Finally, for the third integrable case with $b = g$ we immediately obtain the $Z(t)$ and $N(E)$ from equations (21) and (22) using $F\left(\frac{1}{2}, \frac{1}{2}; 1, 0\right) = 1$.

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